The lifting scheme of 4-channel orthogonal wavelet transforms*

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Abstract The 4-channel smooth wavelets with linear phase and orthogonality are designed from the 2-channel orthogonal wavelets with high transfer vanishing moments. Reversely, for simple lifting scheme of such 4-channel orthogonal wavelet transforms, a new 2-channel orthogonal wavelet associated with this 4-channel wavelet is constructed. The new 2-channel wavelet has at least the same number of vanishing moments as the associated 4-channel one. Finally, by combining the two such 2-channel wavelet systems, the lifting scheme of 4-channel orthogonal wavelet transform, which has simple structure and is easy to apply, is presented.

Keywords: lifting scheme, 4-channel orthogonal wavelet, 2-channel orthogonal wavelet, vanishing moment, transfer vanishing moment, linear phase.

The lifting scheme has been used in both design[1,2] and factorization[3] of filter banks, which is reversible and adopted in-place computation[3]. The lifting scheme can realize fast implementations of the discrete wavelet transform, and design transform that map integers to integers[4,5]. Chen et al.[6] extended Sweldens' conventional 2-channel lifting scheme and presented the M-channel lifting scheme factorization of perfect reconstruction filter banks. However, this M-channel lifting scheme is complicated and difficult to apply. The aim of this paper is to simplify the lifting scheme of a special kind of 4-channel wavelet transforms.

In order to construct Heisenberg FIR (finite implush response) filter banks, Jawerth and Peng[7] proposed a new method of designing 4-channel symmetric low-pass filters. Peng et al.[8] extended that method to design 2^n-channel wavelet systems with beautiful structure from the 2-channel ones. However, the vanishing moments of the designed 4-channel wavelets are incompatible with that of the associated 2-channel ones (see Fig. 1 in Ref. [8]), which is an obstacle to further study of such 4-channel wavelet transforms.

In this paper, the incompatibility is proved theoretically. Then, the concept of transfer vanishing moments of 2-channel wavelets is introduced. Hence, the 2-channel orthogonal wavelet with P transfer vanishing moments will directly generate a symmetric and orthogonal 4-channel wavelet with P vanishing moments. To simplify the lifting scheme of such 4-channel wavelet transform, a new 2-channel orthogonal wavelet is constructed, which has at least the same vanishing moments as the 4-channel wavelet. Finally, the 4-channel orthogonal wavelet transform is divided into two independent 2-channel transforms. By combining such 2-channel wavelet transforms, the lifting scheme of 4-channel orthogonal wavelet transform is presented.

1 Relationship between 2-channel wavelets and 4-channel wavelets

A 2-channel low-pass FIR filter is denoted by $\{f_k\}$. Let $z = e^{i\omega}$, then its z-transform is given by $F(z) = \sum_{k\in Z} f_k z^k$. Its polyphase decomposition is $F(z) = F_0(z^2) + zF_1(z^2)$, with $F_i(z) = \sum_{k\in Z} f_{2k+i} z^k$ $(i = 0, 1)$ being its two polyphase components.

Similarly, suppose the 4-channel low-pass filter is denoted by $\{h_k\}$, and its z-transform is $H(z) = \sum_{k\in Z} h_k z^k$. The polyphase decomposition of h is $H(z) = H_0(z^4) + zH_1(z^4) + z^2H_2(z^4) + z^3H_3(z^4)$, with $H_i(z) = \sum_{k\in Z} h_{4k+i} z^k$, $(i = 0, 1, 2, 3)$ being its polyphase components. Denote by $\{g_k^i\}$ the high-pass filters for $(i = 1, 2, 3)$, and $G^i(z)$ their z-transforms.

We firstly recall the algorithm in Ref. [8].

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Lemma 1 (Theorem 1 [8]). Suppose $f=(f_0, f_1, f_2, \cdots, f_{2L-1})$, and its polyphase decomposition is $F(z)=F_0(z^2)+zF_1(z^2)$. Define $H(z)=F_0(z^4)+zF_1(z^4)+z^{4L-1}(F_0(z^{-4})+z^{-1}F_1(z^{-4}))$. Then f is a 2-channel low-pass filter with length $2L$, if and only if $H(z)$ determines a 4-channel symmetric low-pass filter with length $4L$.

From Lemma 1, we know $F_i(z)=\sum_{k=0}^{L} f_{2k+i}z^k$ for $i=0, 1$, and the polyphase components of the corresponding 4-channel low-pass filter are $H_i(z)=F_i(z)$, $H_{3-i}(z)=z^{L-1}F_i(z^{-1})$ with $i=0, 1$. For the 4-channel system here is orthogonal, its polyphase matrix $\boldsymbol{H}_{\mathrm{p}}(z)$[8] should be paraunitary. Next, the simple version of $\boldsymbol{H}_{\mathrm{p}}(z)$[8] is presented by proper phase-shift.

Theorem 1. Suppose that $f=(f_0, f_1, f_2, \cdots, f_{2L-1})$ is a 2-channel filter with length $2L$. Its two polyphase components are $F_0(z)$ and $F_1(z)$. When L is odd, denote $H_0(z)=z^{-\frac{L-1}{2}}F_0(z)$ and $H_1(z)=z^{-\frac{L-1}{2}}F_1(z)$. And when L is even, denote $H_0(z)=z^{-\frac{L-2}{2}}F_0(z)$ and $H_1(z)=z^{-\frac{L-2}{2}}F_1(z)$. Then $H(z)=H_0(z^4)+zH_1(z^4)+z^2H_1(z^{-4})+z^3H_0(z^{-4})$ determines a 4-channel symmetric orthogonal low-pass filter with length $4L$, if and only if f is a 2-channel orthogonal low-pass filter.

By Theorem 1, we know that when L is either odd or even, the polyphase matrix $\boldsymbol{H}_{\mathrm{p}}$ can be written as

$$\boldsymbol{H}_{\mathrm{p}}=\begin{bmatrix} H_0(z) & H_1(z) & H_1(z^{-1}) & H_0(z^{-1}) \\ -H_0(z) & H_1(z) & -H_1(z^{-1}) & H_0(z^{-1}) \\ -H_1(z) & -H_0(z) & H_0(z^{-1}) & H_1(z^{-1}) \\ H_1(z) & -H_0(z) & -H_0(z^{-1}) & H_1(z^{-1}) \end{bmatrix}, \tag{1}$$

where four entries in each row exactly correspond to the polyphase components of $H(z)$ and $G^i(z)$, $i=1,2,3$. $\boldsymbol{H}_{\mathrm{p}}(z)$ can be proved to be paraunitary and it can determine a 4-channel orthogonal system with beautiful structure.

Theorem 2. Suppose that the 4-channel polyphase decomposition of h is $H(z)=H_0(z^4)+zH_1(z^4)+z^2H_1(z^{-4})+z^3H_0(z^{-4})$, while that of f is denoted by $F(z)=H_0(z^2)+zH_1(z^2)$. If f satisfies $P>1$ vanishing moments, then the number of vanishing moment of h is less than 2.

Proof. If f satisfies $P>1$ vanishing moments, at least $P=2$, then by Ref. [10], $\left.\frac{\mathrm{d}F}{\mathrm{d}\omega}\right|_{\omega=\pi}=0$ holds. Expanding it, we have

$$\frac{\mathrm{d}F}{\mathrm{d}\omega}=2\mathrm{i}z^2\left\{\frac{\mathrm{d}H_0}{\mathrm{d}z}(z^2)+z\frac{\mathrm{d}H_1}{\mathrm{d}z}(z^2)\right\}+\mathrm{i}zH_1(z^2).$$

Let $H_i^1(z)=4z\frac{\mathrm{d}H_i}{\mathrm{d}z}(z)$ for $i=0, 1$ and $\omega=\pi$. Then we have $H_0^1(1)-H_1^1(1)=2$. Suppose that h has $P=2$ vanishing moments, then $\left.\frac{\mathrm{d}H}{\mathrm{d}\omega}\right|_{\omega=\left\{\frac{\pi}{2},\pi,\frac{3\pi}{2}\right\}}=0$ holds. Expanding it, we have

$$\begin{aligned}\frac{\mathrm{d}H}{\mathrm{d}\omega}=\mathrm{i}\{&[H_0^1(z^4)+zH_1^1(z^4)\\ &-z^2H_1^1(z^{-4})-z^3H_0^1(z^{-4})]\\ &+[zH_1(z^4)+2z^2H_1(z^{-4})\\ &+3z^3H_0(z^{-4})]\}.\end{aligned}$$

Let $\omega\in\left\{\frac{\pi}{2},\pi,\frac{3\pi}{2}\right\}$, then $H_0^1(1)=\frac{3}{2}$ and $H_1^1(1)=\frac{1}{2}$ hold. It is in contradiction. Therefore, if f has arbitrary $P>1$ vanishing moments, then h satisfies at most $P=1$ vanishing moment. Q.E.D.

Theorem 2 shows that the vanishing moment of h is incompatible with that of f. For designing smooth 4-channel wavelets, new conditions should be imposed on $F(z)$. Hence, the concept of transfer vanishing moment is introduced. Using the definition in the proof of Theorem 2, let $H_i^1(z)=4z\frac{\mathrm{d}H_i}{\mathrm{d}z}(z)$ with $i=0, 1$. If $F(z)=H_0(z^2)+zH_1(z^2)$ satisfies the equations corresponding to $p=0,1,\cdots,P-1$ listed in Table 1, then $F(z)$ is called satisfying P transfer vanishing moments.

Table 1. Transfer vanishing moments

p	H_0^p	H_1^p
0	$H_0^0(1)=1$	$H_1^0(1)=1$
1	$H_0^1(1)=\frac{3}{2}$	$H_1^1(1)=\frac{1}{2}$
2	$H_0^2(1)=H_1^2(1)+2$	——
3	$H_0^3(1)=\frac{9}{2}H_0^1(1)-\frac{27}{4}$	$H_1^3(1)=\frac{3}{2}H_1^2(1)-\frac{1}{4}$
4	$H_0^4(1)=H_1^4(1)+12H_1^2(1)+2$	——
…	…	…

Theorem 3. Suppose that $H_i(z)$ $(i=0, 1)$ is defined in Theorem 1. $H(z)=H_0(z^4)+zH_1(z^4)+z^2H_1(z^{-4})+z^3H_0(z^{-4})$ determines 4-channel

wavelets with P vanishing moments, if and only if $F(z)=H_0(z^2)+zH_1(z^2)$ determines 2-channel wavelets with P transfer vanishing moments.

To construct 4-channel orthogonal wavelet with P vanishing moments, it is only necessary to design a 2-channel orthogonal wavelet that has P transfer vanishing moments. After the smooth 4-channel wavelets are designed, we then should simplify their transforms. Since the transfer vanishing moment is much different from the traditional vanishing moment, we introduce a new 2-channel low-pass filter $\tilde{f}$ associated with h. It can be proved that this filter has the same vanishing moments as h. The polyphase decomposition of $\tilde{f}$ is

$$\tilde{F}(z) \triangleq H_0(z^2)+zH_1(z^{-2}). \tag{2}$$

Clearly, (i) $\tilde{F}(z)$ is the combination of the first component with the third one of $H(z)$; (ii) $\tilde{F}(z)$ also determines the 2-channel orthogonal filter $\tilde{f}$; (iii) $\tilde{f}_k$, each entry of $\tilde{f}$, corresponds to h_{2k}.

In the following part, we will discuss the compatibility of the 2-channel and 4-channel systems in the vanishing moments.

As for the 2-channel low-pass filter $F(z)=F_0(z^2)+zF_1(z^2)$, we have

Lemma 2. Let $F_i^0(z)=F_i(z)$ $(i=0,1)$, and $F_i^n(z)=2z\dfrac{\mathrm{d}F_i^{n-1}}{\mathrm{d}z}(z)$ $(i=0,1.\ n=1,2,\cdots)$. Then the following equation holds:

$$\begin{aligned}\frac{\mathrm{d}^nF}{\mathrm{d}\omega^n}=\mathrm{i}^n\Big\{&[F_0^n(z^4)+zF_1^n(z^4)]\\&+\binom{n}{1}[zF_1^{n-1}(z^4)]+\cdots\\&+\binom{n}{m}[zF_1^{n-m}(z^4)]+\cdots\\&+\binom{n}{n}[zF_1^0(z^4)]\Big\}.\end{aligned} \tag{3}$$

Similarly, as for the general 4-channel low-pass filter $H(z)=H_0(z^4)+zH_1(z^4)+z^2H_2(z^4)+z^3H_3(z^4)$, we have

Lemma 3. Let $H_i^0(z)=H_i(z)$ $(i=0,1,2,3)$, and $H_i^n(z)=4z\dfrac{\mathrm{d}H_i^{n-1}}{\mathrm{d}z}(z)$ $(i=0,1,2,3.\ n=1,2,\cdots)$. Then

$$\begin{aligned}\frac{\mathrm{d}^nH}{\mathrm{d}\omega^n}=\mathrm{i}^n\Big\{&[H_0^n(z^4)+zH_1^n(z^4)\\&+z^2H_2^n(z^4)+z^3H_3^n(z^4)]\\&+\binom{n}{1}[zH_1^{n-1}(z^4)+2z^2H_2^{n-1}(z^4)\\&+3z^3H_3^{n-1}(z^4)]+\cdots\\&+\binom{n}{m}[zH_1^{n-m}(z^4)+2^mz^2H_2^{n-m}(z^4)\\&+3^mz^3H_3^{n-m}(z^4)]+\cdots\\&+\binom{n}{n}[zH_1^0(z^4)+2^nz^2H_2^0(z^4)\\&+3^nz^3H_3^0(z^4)]\Big\}.\end{aligned} \tag{4}$$

For the special relation between $F_i(z)$ and $H_i(z)$ in Eq. (2), substituting $\tilde{F}(z)$ into Eq. (3) and Eq. (4), we have

$$\left.\frac{\mathrm{d}^n\tilde{F}}{\mathrm{d}\omega^n}\right|_{\omega=\pi}=\frac{1}{2^n}\left(\left.\frac{\mathrm{d}^nH}{\mathrm{d}\omega^n}\right|_{\omega=\frac{\pi}{2}}+\left.\frac{\mathrm{d}^nH}{\mathrm{d}\omega^n}\right|_{\omega=\frac{3\pi}{2}}\right).$$

Clearly, if h satisfies $\forall\, P$ vanishing moments, $\tilde{f}$ corresponding to Eq. (2) satisfies at least P vanishing moments.

Fig. 1 shows the comparison of scaling functions. Fig. 1(a) is the scaling function Φ corresponding to the 4-channel low-pass filter h with length 20 designed in Ref. [8], while Fig. 1(b) and Fig. 1(c) show the scaling functions ϕ and $\tilde{\phi}$ corresponding to f and $\tilde{f}$, respectively. Clearly, $\tilde{\phi}$ is smooth, but ϕ is not.

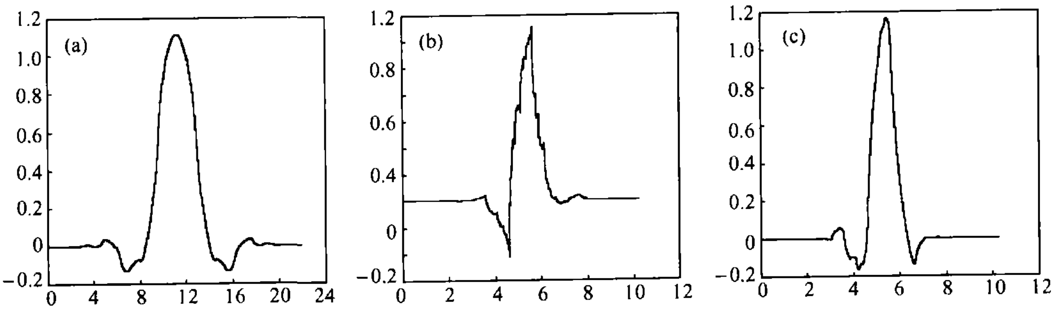

Fig. 1. Comparison of scaling functions.

2 4-channel lifting scheme with simple structure

Because $\boldsymbol{H}_p(z)$ is paraunitary, the 4-channel wavelet system determined by Eq. (1) is orthogonal. As for an input signal x, $X(z)=X_0(z^4)+zX_2(z^4)+z^2X_1(z^4)+z^3X_3(z^4)$ is its polyphase decomposition, where $X_i(z)$ with $i=0, 1, 2, 3$ are its polyphase components. Letting $\boldsymbol{X}(z)=[X_0(z), X_2(z), X_1(z), X_3(z)]^{\mathrm{T}}$, the 4-channel wavelet transform can be expressed by $\boldsymbol{H}_p(z)\boldsymbol{X}(z)$.

Transposing the second column with the third one of $\boldsymbol{H}_p$, we have

$$\widetilde{\boldsymbol{H}}_p=\left[\begin{array}{cc|cc} H_0(z) & H_1(z^{-1}) & H_1(z) & H_0(z^{-1}) \\ -H_0(z) & -H_1(z^{-1}) & H_1(z) & H_0(z^{-1}) \\ \hline -H_1(z) & H_0(z^{-1}) & -H_0(z) & H_1(z^{-1}) \\ H_1(z) & -H_0(z^{-1}) & -H_0(z) & H_1(z^{-1}) \end{array}\right]. \tag{5}$$

For clear discussion, $\widetilde{\boldsymbol{H}}_p$ is written as the form of blocked matrix. Then, the 4-channel wavelet transform becomes

$$\begin{aligned}&\boldsymbol{H}_p(z)\boldsymbol{X}(z)\\ &\quad=\widetilde{\boldsymbol{H}}_p(z)\cdot[X_0(z)X_2(z)X_1(z)X_3(z)]^{\mathrm{T}}.\end{aligned} \tag{6}$$

In the expression of $\widetilde{\boldsymbol{H}}_p$, the two entries in each row of each block correspond to the two polyphase components of one 2-channel filter, either low-pass or high-pass. If the 2-channel polyphase decomposition of the input signal x is written as $X(z)=X^0(z^2)+zX^1(z^2)$, then $X^0(z)=X_0(z^2)+zX_2(z^2)$ and $X^1(z)=X_1(z^2)+zX_3(z^2)$ hold. Therefore, Eq. (6) can be rewritten as

$$\boldsymbol{H}_p(z)\boldsymbol{X}(z)=\left[\begin{array}{c|c} (\downarrow 2)\widetilde{F}(z) & (\downarrow 2)\bar{F}(z) \\ (\downarrow 2)(-\widetilde{F}(z)) & (\downarrow 2)\bar{F}(z) \\ \hline (\downarrow 2)\widetilde{G}(z) & (\downarrow 2)\bar{G}(z) \\ (\downarrow 2)(-\widetilde{G}(z)) & (\downarrow 2)\bar{G}(z) \end{array}\right] \cdot\begin{bmatrix} X^0(z) \\ X^1(z)\end{bmatrix}, \tag{7}$$

where $\widetilde{F}(z)$ is exactly defined by Eq. (2), and $\bar{F}=H_1(z^2)+zH_0(z^{-2})$, while $\widetilde{G}(z)$ and $\bar{G}(z)$ are the high-pass filters corresponding to $\widetilde{F}(z)$ and $\bar{F}(z)$, respectively. Also, $\bar{F}(z)=z^{-1}\widetilde{F}(z^{-1})$ holds, which shows $\tilde{\phi}$ and $\bar{\phi}$ have the same regularity.

From Eq. (7), the 4-channel orthogonal wavelet transform can be simply divided into two independent 2-channel wavelet transforms. Fig. 2 shows two equal structures of 4-channel wavelet transforms. Fig. 2(a) gives the general 4-channel wavelet transform, with $H(z)$ and $G^i(z)$ being the low-pass filter and three high-pass filters for $i=1, 2, 3$. Fig. 2(b) presents the combination of two 2-channel wavelet transforms in Eq. (7), each of which is shown in one dash frame respectively.

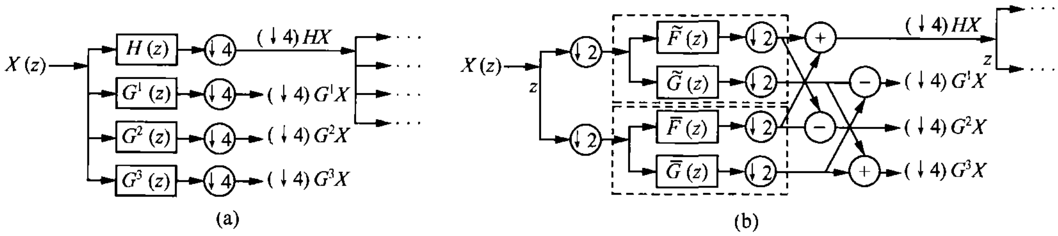

Fig. 2. Two equal structures of 4-channel wavelet transforms.

The upper dash frame in Fig. 2(b) shows the 2-channel orthogonal wavelet transform based on $\widetilde{F}(z)$ and $\widetilde{G}(z)$, and the lower one shows that based on $\bar{F}(z)$ and $\bar{G}(z)$. Therefore, lifting scheme of 4-channel system can be simply divided into two 2-channel lifting schemes. The lifting scheme of Eq. (7) is shown in Fig. 3, where $\bar{s}_i(z)$, $\bar{t}_i(z)$, $\tilde{s}_i(z)$ and $\tilde{t}_i(z)$ represent the lifting steps of the two 2-channel systems, respectively.

Using the factorization algorithm in Ref. [3], we give the factorization of $\widetilde{F}(z)$, whose scaling function is shown in Fig. 1(c):

$$\begin{aligned}\begin{bmatrix} H_0(z) \\ H_1(z^{-1})\end{bmatrix}&=\begin{bmatrix} 1 & \alpha_{00} \\ 0 & 1\end{bmatrix}\cdot\begin{bmatrix} 1 & 0 \\ \alpha_{10}+\alpha_{11}z^{-1} & 1\end{bmatrix}\\ &\cdot\begin{bmatrix} 1 & \alpha_{20}+\alpha_{21}z \\ 0 & 1\end{bmatrix}\cdot\begin{bmatrix} 1 & 0 \\ \alpha_{30}+\alpha_{31}z^{-1} & 1\end{bmatrix}\\ &\cdot\begin{bmatrix} 1 & \alpha_{40}+\alpha_{41}z \\ 0 & 1\end{bmatrix}\cdot\begin{bmatrix} 0 \\ K\end{bmatrix}.\end{aligned} \tag{8}$$

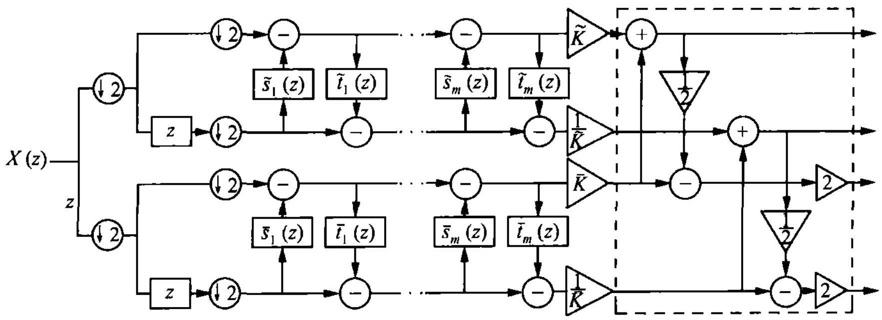

Fig. 3. 4-channel lifting scheme with simple structure.

The parameters are (α_{00}, α_{10}, α_{11}, α_{20}, α_{21}, α_{30}, α_{31}, α_{40}, α_{41}) = (−5.3836482, −.009215727, 0.1795527, −28.043056, 7.6974209, 0.0022974888, 0.017948667, −18.39089, −7.1095098, −0.12772164).

Ref.[4] gives two solutions to convert the scaling operation (K or $\frac{1}{K}$ in Fig.3) into maps from integers into integers. And the operation in the dash frame in Fig.3 can also map integers into integers. Therefore, 4-channel lifting scheme can also be converted into the map from integers into integers.

3 Conclusion

In this paper, smooth symmetric 4-channel orthogonal wavelets have been designed by finding the 2-channel orthogonal wavelets that satisfy high transfer vanishing moments. The concept of transfer vanishing moment is introduced, because the vanishing moment of the 2-channel wavelets is incompatible with that of the 4-channel ones. For simple lifting scheme of such 4-channel orthogonal wavelet transforms, a new 2-channel orthogonal wavelet is constructed, which satisfies at least the same vanishing moment as the 4-channel wavelet. By combining such 2-channel wavelet transforms, a simplified lifting scheme of 4-channel orthogonal transform is given.

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